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ORIGINAL ARTICLE

Model and extended Kuhn–Tucker approach for bilevel multi-follower decision making in a referential-uncooperative situation

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Abstract When multiple followers are involved in a bilevel decision problem, the leader's decision will be affected, not only by the reactions of these followers, but also by the relationships among these followers. One of the popular situations within this bilevel multi-follower issue is where these followers are uncooperatively making their decisions while having cross reference to decision information of the other followers. This situation is called a referential-uncooperative situation in this paper. The well-known Kuhn–Tucker approach has been previously successfully applied to a one-leader-and-one-follower linear bilevel decision problem. This paper extends this approach to deal with the above-mentioned linear referential-uncooperative bilevel multi-follower decision problem. The paper first presents a decision model for this problem. It then proposes an extended Kuhn–Tucker approach to solve this problem. Finally, a numerical example illustrates the application of the extended Kuhn–Tucker approach.

Keywords Bilevel programming · Bilevel multi-follower decision · Kuhn–Tucker approach · Optimization

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1 Introduction

In a bilevel programming (BLP) problem, the leader cannot completely control his/her follower but is influenced by the reaction of his/her follower. Such a situation occurs in decision making of many decentralized organizations. The BLP was motivated by the game theory of Stackelberg [1] in the context of unbalanced economic markets [2]. There have been nearly two dozen algorithms, such as, the *K*th best approach [3,4], Kuhn–Tucker approach [5–7], complementarity pivot approach [8], penalty function approach [9–13], proposed for solving linear BLP problems since the field caught the attention of researchers in the mid-1970s. The Kuhn–Tucker approach has proved to be a valuable analysis tool with a wide range of successful applications in this field [2,6,7,14–16].

Although much research has been carried out, existing bilevel technology has mainly been limited to a specific situation comparing one leader and one follower. However, in a real-world bilevel decision problem, the lower level of a bilevel decision may involve multiple decision units. For example, the dean of a faculty is the leader, and all the heads of departments in the faculty are the followers in making a faculty annual budget. The leader's (the dean's, e.g.) decision will be affected, not only by the reactions of the multiple followers (these heads of departments in the faculty), but also by the relationships among these followers. Each of the leader's possible decisions is influenced by the various reactions of his/her followers who may have had a share in decision information, objectives, and constraints. Hence, a bilevel multi-follower (BLMF) decision problem occurs commonly in any organizational decision practice, and involves various different decision situations. Different relationships among these followers will lead to different bilevel decision situations.

Our previous work [17–19] extended existing linear BLP theory and approaches in one-leader-and-one-follower situation. Following that, we have recently generalized a framework for BLMF decision problems, and identified nine main kinds of relationships amongst these followers [20]. The uncooperative model describes the most popular situation of the nine for BLMF decision problems. This model handles the relationship in which there is no shared decision variable among the followers. Under this uncooperative model, the most basic situation is that any follower also doesn't make any reference to any of the other followers' decisions. A model and related approaches in finding an optimal solution for this particular decision situation have been developed, the reader is referred to [20,21]. An alternative uncooperative situation occurs when despite the fact that the followers are uncooperative in that there is no sharing of decision variables, they do however cross reference information by considering other followers' decision results in each of their own decision objectives and constraints. We call this case as a referential-uncooperative situation, and this paper will particularly focus on this situation.

Following the introduction, this paper proposes a model for linear BLMF decision making in a referential-uncooperative situation in Sect. 2. An extended Kuhn–Tucker approach for solving this model is presented in Sect. 3. A numeric example for this approach is illustrated in Sect. 4. Concluding remarks are given in Sect. 5.

2 The linear BLMF decision model in a referential-uncooperative situation

Under the BLMF framework, if two followers do not have any shared decision variable, it is called an uncooperative relationship. But if one of them has a reference to another follower's decision information in his/her objective or constraint, the two followers are defined as having a referential-uncooperative relationship. When there is a referential-uncooperative relationship in a BLMF decision model, this model is called a referential-uncooperative BLMF decision model. We present this model as follows.

For $x \in X \subset R^n, y_i \in Y_i \subset R^{m_i}$, $F : X \times Y_1 \times \cdots \times Y_K \rightarrow R^1$, and $f_i : X \times Y_1 \times \cdots \times Y_K \rightarrow R^1$, $i = 1, 2, \ldots, K$, a linear BLMF decision problem where $K(\geqslant 2)$ followers are involved and there are no shared decision variables, but shared information in objective functions and constraint functions among the followers which is defined as follows:

$$\min_{x \in X} F(x, y_1, \ldots, y_K) = cx + \sum_{s=1}^{K} d_s y_s, \tag{1a}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \leqslant b, \tag{1b}$$

$$\min_{y_i \in Y_i} f_i(x, y_1, \ldots, y_K) = c_i x + \sum_{s=1}^{K} e_{is} y_s, \tag{1c}$$

$$\text{subject to } A_i x + \sum_{s=1}^{K} C_{is} y_s \leqslant b_i, \tag{1d}$$

where $c \in R^n$, $c_i \in R^n$, $d_i \in R^{m_i}$, $e_{is} \in R^{m_s}$, $b \in R^p$, $b_i \in R^{q_i}$, $A \in R^{p\times n}$, $B_i \in R^{p\times m_i}$, $A_i \in R^{q_i \times n}$, $C_{is} \in R^{q_i \times m_s}$, $i, s = 1, 2, \ldots, K$.

To find an optimal solution for this model we introduce definitions of constraint region, projection of S onto the leader's decision space, feasible set for each follower, and inducible region for a linear BLMF decision problem.

Definition 1

(a) Constraint region of a linear BLMF decision problem:

$$S = \left\{ (x, y_1, \ldots, y_K) \in X \times Y_1 \times \cdots \times Y_k, Ax + \sum_{s=1}^{K} B_s y_s \leqslant b, \right.$$
$$\left. A_i x + \sum_{s=1}^{K} C_{is} y_s \leqslant b_i, i = 1, 2, \ldots, K \right\}.$$

The constraint region refers to all possible combinations of choices that the leader and followers may make.

(b) Projection of S onto the leader's decision space:

$$S(X) = \left\{ x \in X : \exists y_i \in Y_i, Ax + \sum_{s=1}^{K} B_s y_s \leqslant b, A_i x + \sum_{s=1}^{K} C_{is} y_s \leqslant b_i, i = 1, 2, \ldots, K \right\}.$$

(c) Feasible set for each follower $\forall x \in S(X)$:

$$S_i(x) = \{y_i \in Y_i : (x, y_1, \ldots, y_K) \in S\}.$$

The feasible region for each follower is affected by the leader's choice of x, and the allowable choices of each follower are the elements of S.

(d) Each follower's rational reaction set for $x \in S(X)$:
$P_i(x) = \{y_i \in Y_i : y_i \in \arg\min[f_i(x, \hat{y}_i, y_j, j = 1, 2, \ldots, K, j \neq i) : \hat{y}_i \in S_i(x)]\}$, where $i = 1, 2, \ldots, K$, $\arg\min[f_i(x, \hat{y}_i, y_j, j = 1, 2, \ldots, K, j \neq i) : \hat{y}_i \in S_i(x)]$ $= \{y_i \in S_i(x) : f_i(x, y_1, \ldots, y_K) \leqslant f_i(x, \hat{y}_i, y_j, j = 1, 2, \ldots, K, j \neq i), \hat{y}_i \in S_i(x)\}$. The followers observe the leader's action and simultaneously react by selecting y_i from their feasible set to minimize their objective function.

(e) Inducible region:

$$\mathrm{IR} = \{(x, y_1, \ldots, y_K) : (x, y_1, \ldots, y_K) \in S, y_i \in P_i(x), i = 1, 2, \ldots, K\}.$$

Thus the model given by expressions (1a)–(1d) can be rewritten in terms of the above notations as follows

$$\min\{F(x, y_1, \ldots, y_K) : (x, y_1, \ldots, y_K) \in \mathrm{IR}\}. \tag{2}$$

We propose the following theorem to characterize the condition under which there exists an optimal solution for a referential-uncooperative linear BLMF decision problem shown in (1a)–(1d).

Theorem 1 *If S is nonempty and compact, there exists an optimal solution for a linear BLMF decision problem.*

Proof Since S is nonempty, there exist a point $(x^*, y_1^*, \ldots, y_K^*) \in S$. Then, we have

$$x^* \in S(X) \neq \phi$$

by Definition 1(b). Consequently, we have

$$S_i(x^*) \neq \phi, \quad i = 1, 2, \ldots, K$$

by Definition 1(c). Because S is compact and Definition 1(d), we have

$$\begin{aligned} P_i(x^*) &= \{y_i \in Y_i : y_i \in \arg\min[f_i(x^*, \hat{y}_i, y_j, j = 1, 2, \ldots, K, j \neq i) : \hat{y}_i \in S_i(x^*)]\} \\ &= \{y_i \in Y_i : y_i \in \{y_i \in S_i(x^*) : \\ &\quad f_i(x^*, y_1, \ldots, y_K) \leqslant f_i(x^*, \hat{y}_i, y_j, j = 1, 2, \ldots, K, j \neq i), \hat{y}_i \in S_i(x^*)\}\} \neq \phi, \end{aligned}$$

where $i = 1, 2, \ldots, K$. Hence, there exists $y_i^0 \in P_i(x^*)$, $i = 1, 2, \ldots, K$ such that $(x^*, y_1^0, \ldots, y_K^0) \in S$. Therefore, we have

$$\mathrm{IR} = \{(x, y_1, \ldots, y_K) : (x, y_1, \ldots, y_K) \in S, y_i \in P_i(x), i = 1, 2, \ldots, K\} \neq \phi$$

by Definition 1(e). Because we are minimizing a linear function

$\min_{x \in X} F(x, y_1, \ldots, y_K) = cx + \sum_{s=1}^{K} d_s y_s$ over IR, which is nonempty and bounded, an optimal solution to the linear BLMF decision problem must exist. The proof is completed. □

3 An extended Kuhn–Tucker approach for referential-uncooperative BLMF decision problem

In this section, we extend the well known Kuhn–Tucker conditions so as to provide necessary and sufficient conditions for the linear referential-uncooperative BLMF decision problem defined by expression (1a–1d).

Let write a linear programming (LP) as follows.

$$\begin{aligned} &\min f(x) = cx, \\ &\text{subject to } Ax \geqslant b, \\ &\qquad\qquad x \geqslant 0, \end{aligned}$$

where c is an n-dimensional row vector, b an m-dimensional column vector, A an $m \times n$ matrix with $m \leqslant n$, and $x \in R^n$.

Let $\lambda \in R^m$ and $\mu \in R^n$ be the dual variables associated with constraints $Ax \geqslant b$ and $x \geqslant 0$, respectively. Then we note that Bard [2] gave the following proposition.

Proposition 1 A necessary and sufficient condition that (x^*) solves above LP is that there exist (row) vectors λ^*, μ^* such that (x^*, λ^*, μ^*) solves:

$$\begin{aligned} &\lambda A - \mu = -c, \\ &Ax - b \geqslant 0, \\ &\lambda(Ax - b) = 0, \\ &\mu x = 0, \\ &x \geqslant 0, \;\; \lambda \geqslant 0, \;\; \mu \geqslant 0. \end{aligned}$$

The proof for this proposition is given in [2, pp. 59–60]. We will next utilize this to derive the extended Kuhn–Tucker approach for referential-uncooperative BLMF decision problem.

Let $u_i \in R^p$, $v_i \in R^{q_1+q_2+\cdots+q_K}$ and $w_i \in R^{m_i} (i = 1, 2, \ldots, K)$ be the dual variables associated with constraints $(Ax + \sum_{s=1}^{K} B_s y_s \leqslant b)$, $(A'x + \sum_{s=1}^{K} C'_s y_s \leqslant b')$, and $y_i \geqslant 0 \quad (i = 1, \ldots, K)$, respectively, where $A' = (A_1, A_2, \ldots, A_K)^T$, $C'_i = (C_{i1}, C_{i2}, \ldots, C_{iK})^T$, and $b' = (b_1, b_2, \ldots, b_K)^T$.

We have the following theorem, which gives the necessary and sufficient conditions for solving a referential-uncooperative BLMF decision problem.

Theorem 2 *A necessary and sufficient condition that* $(x^*, y_1^*, \ldots, y_K^*)$ *solves the linear BLMF decision problem* (1a)–(1d) *is that there exist (row) vectors* $u_1^*, u_2^*, \ldots, u_K^*$, $v_1^*, v_2^*, \ldots, v_K^*$, *and* $w_1^*, w_2^*, \ldots, w_K^*$ *such that* $(x^*, y_1^*, \ldots, y_K^*, u_1^*, \ldots, u_K^*, v_1^*, \ldots, v_K^*, w_1^*, \ldots, w_K^*)$ *solves:*

$$\min_{x \in X} F(x, y_1, \ldots, y_K) = cx + \sum_{s=1}^{K} d_s y_s, \tag{3a}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \leqslant b, \tag{3b}$$

$$A'x + \sum_{s=1}^{K} C'_s y_s \leqslant b', \tag{3c}$$

$$u_iB_i + v_iC_i' - w_i = -e_{ii} \tag{3d}$$

$$u_i\left(b - Ax - \sum_{s=1}^{K} B_sy_s\right) + v_i\left(b' - A'x - \sum_{s=1}^{K} C_s'y_s\right) + w_iy_i = 0, \tag{3e}$$

$$x \geqslant 0,\ \ y_j \geqslant 0,\ \ u_j \geqslant 0,\ \ v_j \geqslant 0,\ \ w_j \geqslant 0,\ \ j = 1, 2, \ldots, K, \tag{3f}$$

where $i = 1, 2 \ldots, K$.

Proof

(1) Let us get an explicit expression of (2).

Rewrite (2) as follows:

$$\begin{aligned} &\min F(x, y_1, \ldots, y_K),\\ &\text{subject to } (x, y_1, \ldots, y_K) \in \mathrm{IR}. \end{aligned}$$

We have

$$\begin{aligned} &\min F(x, y_1, \ldots, y_K),\\ &\text{subject to } (x, y_1, \ldots, y_K) \in S,\\ &\qquad y_i = P_i(x), \end{aligned}$$

where $i = 1, 2, \ldots, K$, by Definition 1(e). Then, we have

$$\begin{aligned} &\min F(x, y_1, \ldots, y_K),\\ &\text{subject to } (x, y_1, \ldots, y_K) \in S,\\ &\qquad y_i \in \arg\min[f_i(x, \hat{y}_i, y_j, j = 1, 2, \ldots, K, j \neq i) : \hat{y}_i \in S_i(x)], \end{aligned}$$

where $i = 1, 2, \ldots, K$, by Definition 1(d). We rewrite it as:

$$\begin{aligned} &\min F(x, y_1, \ldots, y_K),\\ &\text{subject to } (x, y_1, \ldots, y_K) \in S,\\ &\qquad \min f_i(x, y_1, \ldots, y_K),\\ &\qquad \text{subject to } y_i \in S_i(x), \end{aligned}$$

where $i = 1, 2, \ldots, K$. We have

$$\begin{aligned} &\min F(x, y_1, \ldots, y_K),\\ &\text{subject to } (x, y_1, \ldots, y_K) \in S,\\ &\qquad \min_{y_i \in Y_i} f_i(x, y_1, \ldots, y_K),\\ &\qquad \text{subject to } (x, y_1, \ldots, y_K) \in S, \end{aligned}$$

where $i = 1, 2, \ldots, K$, by Definition 1(c). Consequently, we can have

$$\min_{x \in X} F(x, y_1, \ldots, y_K) = cx + \sum_{s=1}^{K} d_sy_s, \tag{4a}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \leqslant b, \tag{4b}$$

$$A_j x + \sum_{s=1}^{K} C_{is} y_s \leqslant b_i, \quad j = 1, 2, \ldots, K, \tag{4c}$$

$$\min_{y_i \in Y_i} f_i(x, y_i, \ldots, y_K) = c_i x + \sum_{s=1}^{K} e_{is} y_s, \tag{4d}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \leqslant b, \tag{4e}$$

$$A_j x + \sum_{s=1}^{K} C_{is} y_s \leqslant b_i, \quad j = 1, 2, \ldots, K, \tag{4f}$$

where $i = 1, 2, \ldots, K$, by Definition 1(a).

This simple transformation has shown that solving the linear BLMF decision problem (1a)–(1d) is equivalent to solving (4a)–(4f).

(2) Hence the proof of necessity is shown by (4a)–(4f).

(3) Sufficiency is proved as follows:

If $(x^*, y_1^*, \ldots, y_K^*)$ is the optimal solution of (1a)–(1d), we need to show that there exist (row) vectors $u_1^*, u_2^*, \ldots, u_K^*, v_1^*, v_2^*, \ldots, v_K^*$, and $w_1^*, w_2^*, \ldots, w_K^*$ such that $(x^*, y_1^*, \ldots, y_K^*, u_1^*, \ldots, u_K^*, v_1^*, \ldots, v_K^*, w_1^*, \ldots, w_K^*)$ to solve (4a)–(4f). Going one step farther, we only need to prove that there exist (row) vectors u_1^*, $u_2^*, \ldots, u_K^*$, v_1^*, $v_2^*, \ldots, v_K^*$, and w_1^*, $w_2^*, \ldots, w_K^*$ such that $(x^*$, $y_1^*, \ldots, y_K^*$, $u_1^*, \ldots, u_K^*$, $v_1^*, \ldots, v_K^*$, $w_1^*, \ldots, w_K^*)$ satisfies the following expressions (5a)–(5d) below:

$$u_i B_i + v_i C_i' - w_i = -e_{ii}, \tag{5a}$$

$$u_i \left(b - Ax - \sum_{s=1}^{K} B_s y_s \right) = 0, \tag{5b}$$

$$v_i \left(b' - A'x - \sum_{s=1}^{K} C_s' y_s \right) = 0, \tag{5c}$$

$$w_i y_i = 0, \tag{5d}$$

where $u_i \in R^p$, $v_i \in R^{q_1+q_2+\cdots+q_K}$, $w_i \in R^{m_i}$, $i = 1, 2, \ldots, K$ and they are not negative variables. Because $(x^*, y_1^*, \ldots, y_K^*)$ is the optimal solution of (1a)–(1d), we have

$$(x^*, y_1^*, \ldots, y_K^*) \in \mathrm{IR},$$

by (2). Thus we have

$$y_i^* \in P_i(x^*),$$

where $i = 1, 2, \ldots, K$, by Definition 1(e). Consequently $(y_1^*, y_2^*, \ldots, y_K^*)$ is the optimal solution to the following problem

$$\min(f_i(x^*, y_1, \ldots, y_K) : y_i \in S_i(x^*)),$$

where $i = 1, 2, \ldots, K$, by Definition 1(d). Rewrite it as follows

$$\begin{aligned} &\min f_i(x, y_1, \ldots, y_K), \\ &\text{subject to } \; y_i \in S_i(x), \\ &\qquad\qquad\quad x = x^*, \\ &\qquad\qquad\quad y_j = y_j^*, \quad j = 1, 2, \ldots, K, \; j \neq i, \end{aligned}$$

where $i = 1, 2, \ldots, K$. From Definition 1(c), we have

$$\min f_i(x, y_1, \ldots, y_K) = c_i x + \sum_{s=1}^{K} e_{is} y_s, \tag{6a}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \leqslant b, \tag{6b}$$

$$A_j x + \sum_{s=1}^{k} C_{js} y_s \leq b_j, \quad j = 1, 2, \ldots, K, \tag{6c}$$

$$x = x^*, \tag{6d}$$

$$y_i \geqslant 0, \tag{6e}$$

$$y_j = y_j^*, \quad j = 1, 2, \ldots, K, \; j \neq i, \tag{6f}$$

where $i = 1, 2, \ldots, K$. Let us define:

$A' = (A_1, A_2, \ldots, A_K)^{-1}$, $b' = (b_1, b_2, \ldots, b_K)^{-1}$, $C'_i = (C_{i1}, \; C_{i2}, \ldots, C_{iK})^{-1}$, $i = 1, 2 \ldots, K$. To simplify (6c), we can have

$$\begin{aligned} &\min f_i(x, y_1, \ldots, y_K) = c_i x + \sum_{s=1}^{K} e_{is} y_s, \\ &\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \leqslant b, \\ &\qquad\qquad A'x + \sum_{s=1}^{K} C'_s y_s \leqslant b', \\ &\qquad\qquad x = x^*, \\ &\qquad\qquad y_i \geqslant 0, \\ &\qquad\qquad y_j = y_j^*, \quad j = 1, 2, \ldots, K, \; j \neq i, \end{aligned}$$

where $i = 1, 2, \ldots, K$.

Thus simplify it, we can have

$$\min f_i(y_i) = e_{ii} y_i, \tag{7a}$$

$$\text{subject to } -\begin{pmatrix} B_i \\ C'_i \end{pmatrix} y_i \geqslant -\begin{pmatrix} b - Ax^* - \sum\limits_{s=1, s\neq i}^{K} B_s y_s^* \\ b'_i - A'x^* - \sum\limits_{s=1, s\neq i}^{K} C'_s y_s^* \end{pmatrix}, \tag{7b}$$

$$y_i \geqslant 0, \tag{7c}$$

where $i = 1, 2, \ldots, K$.

Now we see that y_i^* is the optimal solution of (7a)–(7c) which is a LP problem. By Proposition 1, there exists vectors λ_i^*, μ_i^*, $i = 1, 2, \ldots, K$ that satisfy the system below

$$\lambda_i \begin{pmatrix} B_i \\ C_i' \end{pmatrix} - \mu_i = -e_{ii},$$

$$\lambda_i \left(-\begin{pmatrix} B_i \\ C_i' \end{pmatrix} y_i + \begin{pmatrix} b - Ax^* - \sum\limits_{s=1, s\neq i}^{K} B_s y_s^* \\ b' - A'x^* - \sum\limits_{s=1, s\neq i}^{K} C_s' y_s^* \end{pmatrix} \right) = 0,$$

$$\mu_i y_i = 0,$$

where $\lambda_i \in R^{p+q_1+\ldots+q_K}$, $\mu_i \in R^{m_i}$, $i = 1, 2, \cdots, K$.

Let $u_i \in R^p$, $v_i \in R^{q_i+q_2+\cdots+q_K}$, $w_i \in R^{m_i}$ and define

$$\lambda_i = (u_i, v_i),$$

$$w_i = \mu_i,$$

where $i = 1, 2, \ldots, K$. Thus we have $(x^*, y_1^*, \ldots, y_K^*, u_1^*, \ldots, u_K^*, v_1^*, \ldots, v_K^*, w_1^*, \ldots, w_K^*)$ that satisfy (8a)–(8n). Our proof is completed. □

Theorem 2 means that the most direct approach to solving (1a)–(1d) is to solve the equivalent mathematical program given in (7a)–(7c). One advantage this offers is that it allows a more robust model to be solved without introducing any new computational difficulty.

4 Numerical example for the extended Kuhn–Tucker approach

We apply the above Necessity and Sufficiency conditions given in Theorem 2 to a simple linear referential-uncooperative BLMF decision problem to illustrate how the extended Kuhn–Tucker approach is used.

Example Consider the following linear BLMF decision problem with $x_1, x_2 \in R^1$, $y_1, y_2 \in R^1$, $y_3 \in R^1$and $X = \{x_1 \geqslant 0, x_2 \geqslant 0\}$, $Y = \{y_1 \geqslant 0, y_2 \geqslant 0, y_3 \geqslant 0\}$,

$$\begin{aligned}
&\min_{x_1 \in X, x_2 \in X} F(x_1, x_2, y_1, y_2, y_3) = -8x_1 - 4x_2 - 4y_1 + 40y_2 + 4y_3,\\
&\text{subject to } 2x_1 - y_1 + 2y_2 - 0.5y_3 \leqslant 1,\\
&\qquad \min_{y_1 \in Y} f_1(x_1, x_2, y_1, y_2, y_3) = 2x_1 + x_2 + 2y_1 - y_2 - y_3,\\
&\qquad \min_{y_2 \in Y} f_2(x_1, x_2, y_1, y_2, y_3) = x_1 + 2x_2 - y_1 + 2y_2 - y_3,\\
&\qquad \min_{y_3 \in Y_3} f_3(x_1, x_2, y_1, y_2, y_3) = 3x_1 + 3x_2 + y_1 + y_2 - 2y_3,\\
&\qquad \text{subject to } 2x_2 + 2y_1 - y_2 - 0.5y_3 \leqslant 1,\\
&\qquad\qquad -y_1 + y_2 + y_3 \leqslant 1.
\end{aligned}$$

This is a referential-uncooperative problem. Each of the three followers has an individual objective and decision variable, but considering other followers' decision variables in his/her objective works.

We illustrate the application of the extended Kuhn–Tucker approach to this example. According to the proposed extended Kuhn–Tucker approach, we write all the inequalities excepting $x_1 \geqslant 0, x_2 \geqslant 0$ of the transferred form of the example as follows:

$$\begin{aligned}
&g_{u,1}(x_1,x_2,y_1,y_2,y_3) = 1-(2x_1-y_1+2y_2-0.5y_3) \geqslant 0,\\
&g_{v,1}(x_1,x_2,y_1,y_2,y_3) = 1-(2x_2+2y_1-y_2-0.5y_3) \geqslant 0,\\
&g_{v,2}(x_1,x_2,y_1,y_2,y_3) = 1-(-y_1+y_2+y_3) \geqslant 0,\\
&g_{w1,1}(x_1,x_2,y_1,y_2,y_3) = y_1 \geqslant 0,\\
&g_{w2,1}(x_1,x_2,y_1,y_2,y_3) = y_2 \geqslant 0,\\
&g_{w3,1}(x_1,x_2,y_1,y_2,y_3) = y_3 \geqslant 0.
\end{aligned}$$

From (3a) to (3f), we have

$$\min(-8x_1-4x_2-4y_1+40y_2+4y_3), \tag{8a}$$

$$\text{subject to } 2x_1-y_1+2y_2-0.5y_3 \leqslant 1, \tag{8b}$$

$$2x_2+2y_1-y_2-0.5y_3 \leqslant 1, \tag{8c}$$

$$-y_1+y_2+y_3 \leqslant 1, \tag{8d}$$

$$-u_{11}+2v_{11}-v_{12}-w_{11} = -2, \tag{8e}$$

$$2u_{21}-v_{21}+v_{22}-w_{21} = -2, \tag{8f}$$

$$-0.5u_{31}-0.5v_{31}+v_{32}-w_{31} = 2, \tag{8g}$$

$$g_{u,1}u_{11}+g_{v,1}v_{11}+g_{v,2}v_{12}+g_{w1,1}w_{11} = 0, \tag{8h}$$

$$g_{u,1}u_{21}+g_{v,1}v_{21}+g_{v,2}v_{22}+g_{w2,1}w_{21} = 0, \tag{8i}$$

$$g_{u,1}u_{31}+g_{v,1}v_{31}+g_{v,2}v_{32}+g_{w3,1}w_{31} = 0, \tag{8j}$$

$$x_1 \geqslant 0,\ \ x_2 \geqslant 0,\ \ y_1 \geqslant 0,\ \ y_2 \geqslant 0,\ \ y_3 \geqslant 0, \tag{8k}$$

$$u_{11} \geqslant 0,\ \ v_{11} \geqslant 0,\ \ v_{12} \geqslant 0,\ \ w_{11} \geqslant 0, \tag{8l}$$

$$u_{21} \geqslant 0,\ \ v_{21} \geqslant 0,\ \ v_{22} \geqslant 0,\ \ w_{21} \geqslant 0, \tag{8m}$$

$$u_{31} \geqslant 0,\ \ v_{31} \geqslant 0,\ \ v_{32} \geqslant 0,\ \ w_{31} \geqslant 0. \tag{8n}$$

From (8e), (8f), (8g), (8l), (8m) and (8n), we have following six possibilities.

Case1 $(u_{11},v_{11},v_{12},w_{11},u_{21},v_{21},v_{22},w_{21},u_{31},v_{31},v_{32},w_{31}) = (2,0,0,0,0,2,0,0,0,0,2,0)$.
Case2 $(u_{11},v_{11},v_{12},w_{11},u_{21},v_{21},v_{22},w_{21},u_{31},v_{31},v_{32},w_{31}) = (0,0,2,0,0,2,0,0,0,0,2,0)$.
Case3 $(u_{11},v_{11},v_{12},w_{11},u_{21},v_{21},v_{22},w_{21},u_{31},v_{31},v_{32},w_{31}) = (0,0,0,2,0,2,0,0,0,0,2,0)$.
Case4 $(u_{11},v_{11},v_{12},w_{11},u_{21},v_{21},v_{22},w_{21},u_{31},v_{31},v_{32},w_{31}) = (2,0,0,0,0,0,0,2,0,0,2,0)$.
Case5 $(u_{11},v_{11},v_{12},w_{11},u_{21},v_{21},v_{22},w_{21},u_{31},v_{31},v_{32},w_{31}) = (0,0,2,0,0,0,0,2,0,0,2,0)$.
Case6 $(u_{11},v_{11},v_{12},w_{11},u_{21},v_{21},v_{22},w_{21},u_{31},v_{31},v_{32},w_{31}) = (0,0,0,2,0,0,0,2,0,0,2,0)$.

Table 1 Procedure of solution

Case	Solution occurs at the point	F	f_1	f_2	f_3
2	$(x_1^2, x_2^2, y_1^2, y_2^2, y_3^2) = (1.5, 0, 1, 0, 2)$	$F^2 = -8$	$f_1^2 = 3$	$f_2^2 = -1.5$	$f_3^2 = 1.5$
3	$(x_1^3, x_2^3, y_1^3, y_2^3, y_3^3) = (0.75, 0.75, 0, 0, 1)$	$F^3 = -5$	$f_1^3 = 1.25$	$f_2^3 = 1.25$	$f_3^3 = 2.5$
4	$(x_1^4, x_2^4, y_1^4, y_2^4, y_3^4) = (1.5, 0, 1, 0, 2)$	$F^4 = -8$	$f_1^4 = 3$	$f_2^4 = -1.5$	$f_3^4 = 1.5$
5	$(x_1^5, x_2^5, y_1^5, y_2^5, y_3^5) = (1.5, 0, 1, 0, 2)$	$F^5 = -8$	$f_1^5 = 3$	$f_2^5 = -1.5$	$f_3^5 = 1.5$
6	$(x_1^6, x_2^6, y_1^6, y_2^6, y_3^6) = (0.75, 0.75, 0, 0, 0)$	$F^6 = -5$	$f_1^6 = 1.25$	$f_2^6 = 1.25$	$f_3^6 = 2.5$

From Case 1, (8h), (8i), (8j) and (8k), we have

$$\begin{aligned}
g_{u,1}(x_1, x_2, y_1, y_2, y_3) &= 1 - (2x_1 - y_1 + 2y_2 - 0.5y_3) = 0, \\
g_{v,1}(x_1, x_2, y_1, y_2, y_3) &= 1 - (2x_2 + 2y_1 - y_2 - 0.5y_3) = 0, \\
g_{v,2}(x_1, x_2, y_1, y_2, y_3) &= 1 - (-y_1 + y_2 + y_3) = 0.
\end{aligned}$$

Consequently, (8a)–(8n) can be rewritten as follows:

$$\begin{aligned}
&\min(-8x_1 - 4x_2 - 4y_1 + 40y_2 + 4y_3), \\
&\text{subject to} \quad 2x_1 - y_1 + 2y_2 - 0.5y_3 = 1, \\
&\qquad\qquad\quad 2x_2 + 2y_1 - y_2 - 0.5y_3 = 1, \\
&\qquad\qquad\quad -y_1 + y_2 + y_3 = 1, \\
&\qquad\qquad\quad x_1 \geqslant 0, \;\; x_2 \geqslant 0, \;\; y_1 \geqslant 0, \;\; y_2 \geqslant 0, \;\; z \geqslant 0.
\end{aligned}$$

Using the simplex algorithm [2], we found that a solution occurs at the point $(x_1^1, x_2^1, y_1^1, y_2^1, y_3^1) = (1.5, 0, 1, 0, 2)$ with $F^1 = -8, f_1^1 = 3, f_2^1 = -1.5$, and $f_3^1 = 1.5$.

By using the same approach as that of Case 1, we obtain a solution for each case as shown in Table 1.

By examining above procedure shown in Table 1, we found that the optimal solution for this example which occurs at the point $(x_1^*, x_1^*, y_1^*, y_2^*, y_3^*) = (1.5, 0, 1, 0, 2)$ with $F^* = -8, f_1^* = 3, f_2^* = -1.5, f_3^* = 1.5$.

5 Concluding remark

Different relationships occur among multiple followers in a BLMF decision problem and these can cause multiple different processes for deriving an optimal solution for the upper level's decision making. The referential-uncooperative situation is one of the frequenting occurring cases in BLMF decision practices. For solving such a BLMF decision problem, this paper extended the Kuhn–Tucker approach from dealing with one-leader-and-one-follower to dealing with referential-uncooperative multiple followers. This paper further illustrated the details of the proposed approach by a numerical example. Initial experimental results showed this new extended approach as effective for solving the proposed BLMF decision problem.

Like most really powerful ideas, the basic notion of Nash equilibrium is very simple, even obvious. Its mathematical extensions and implications are not, however. The idea of this natural "sticking point" is that no single player can benefit from unilaterally changing his or her move—a noncooperative best-response equilibrium [22]. In a future research, we will explore how this concept can be applied in our BLMF problem research. Some practical use of this extended approach to show real world problems will also be considered as our future research task for BLMF decision making in the referential-uncooperative situation.

References

1. Stackelberg, H.V.: The Theory of the Market Economy. Oxford University Press, Oxford (1952)
2. Bard, J.: Practical Bilevel Optimization: Algorithms and Applications. Kluwer Academic Publishers, Boston (1998)
3. Candler, W., Townsley, R.: A linear twolevel programming problem. Comput. Oper. Res. **9**, 59–76 (1982)
4. Bialas, W., Karwan, M.: Two-level linear programming. Manage. Sci. **30**, 1004–1020 (1984)
5. Bard, J., Falk, J.: An explicit solution to the multilevel programming problem. Comput. Oper. Res. **9**, 77–100 (1982)
6. Bialas, W., Karwan, M.: Multilevel linear programming. Technical Report 781, State University of New York at Buffalo, (1978)
7. Hansen, P., Jaumard, B., Savard, G.: New branchandbound rules for linear bilevel programming. SIAM J. Sci. Stat. Comput. **13**, 1194–1217 (1992)
8. Bialas, W., Karwan, M., Shaw, J.: A Parametric Complementary Pivot Approach for Twolevel Linear Programming. Technical Report 802, State University of New York at Buffalo, (1980)
9. Aiyoshi, E., Shimizu, K.: Hierarchical decentralized systems and its new solution by a barrier method. IEEE Trans. Sys. Man Cybernet. **11**, 444–449 (1981)
10. White, D., Anandalingam, G.: A penalty function approach for solving bilevel linear programs. J. Global Optim. **3**, 397–419 (1993)
11. Scholtes, S., Stöhr, M.: Exact penalization of mathematical programs with equilibrium costraints. SIAM J. Control Optim. **37**, 617–652 (1999)
12. Marcotte, P., Zhu, D.: Exact and inexact penalty methods for the generalized bilevel programming problem. Math. Program. **74**, 142–157 (1996)
13. Ye, J.J., Zhu, D.L.: Exact penalization and necessary optimality conditions for generalized bilevel programming problems. SIAM J. Optim. **7**, 481–507 (1997)
14. Migdalas, A.: Bilevel programming in traffic planning: models, methods and challenge. J. Global Optim. **7**, 381–405 (1995)
15. Migdalas, A., Pardalos, P.M., Värbrand, P.: Multilevel Optimization-Algorithms and Application. Kluwer Academics, Dordrecht (1997)
16. Marcotte, P.: Network design problem with congestion effects: a case of bilevel programming. Math. Program. **34**, 142–162 (1986)
17. Shi, C., Zhang, G., Lu, J.: On the definition of linear bilevel programming solution. Appl. Math. Comput. **160**, 169–176 (2005)
18. Shi, C., Lu, J., Zhang, G.: An extended *K*th-best approach for linear bilevel programming. Appl. Math. Comput. **164**, 843–855 (2005)
19. Shi, C., Lu, J., Zhang, G.: An extended Kuhn–Tucker approach for linear bilevel programming. Appl. Math. Comput. **162**, 51–63 (2005)
20. Lu, J., Shi, C., Zhang, G.: On bilevel multi-follower decision making: general framework and solutions. Inform. Sci. 176, 1607–1627 (2006)
21. Shi, C., Zhang, G., Lu, J.: A *K*th-best approach for linear bilevel multi-follower programming. J. Global Optim. 33, 563–578 (2005)
22. Ritzberger, K.: Foundations of non-cooperative game theory. Oxford University Press, Oxford (2002)